Assignment 5.

1. (a) Let $\alpha(\theta) = (a\cos\theta, b\sin\theta)$

$\therefore$ the curvature

$$k = \frac{\det(\alpha', \alpha'')}{|\alpha'|^3} = \frac{ab}{[a^2\sin^2\theta + b^2\cos^2\theta]^{\frac{3}{2}}}$$

$$\therefore \quad k|_{(a,0)} = k(0) = \frac{a}{b^2}$$

$$k|_{(0,b)} = k(\tfrac{\pi}{2}) = \frac{b}{a^2}$$

(b) The plane is $(x, r\cos\theta_0, r\sin\theta_0)$ where $(x, r) \in R^2$; $\theta_0 \neq \frac{\pi}{2}$.

$$\therefore \begin{cases} x^2 + y^2 = 1 \\ (x,y,z) \in (x, r\cos\theta_0, r\sin\theta_0) \end{cases} \Rightarrow x^2 + r^2\cos^2\theta_0 = 1$$

$\therefore$ the intersection is an ellipse.

Let the intersection to be

$$\alpha(t) = (\cos t, \sin t, \sin t \cdot \tan\theta_0) \qquad (x = \cos t,\ r = \frac{\sin t}{\cos\theta_0})$$

where $t \in [0, 2\pi)$

$$\therefore \quad \alpha'(t) = (-\sin t, \cos t, \cos t \cdot \tan\theta_0)$$

$$\alpha''(t) = (-\cos t, -\sin t, -\sin t \cdot \tan\theta_0)$$

- the geodesic curvature of α w.r.t the plane

$\because$ $\alpha''(t)$ always lies on the plane

$$\therefore \quad |k_{g,\text{plane}}| = \left| \frac{\alpha' \times \alpha''}{|\alpha'|^3} \right| = \frac{(\tan^2\theta_0 + 1)^{\frac{1}{2}}}{[\sin^2 t + \cos^2 t(1 + \tan^2\theta_0)]^{\frac{3}{2}}}$$

$$\therefore \quad |k_{g,\text{plane}}(0)| = \cos^2\theta_0$$

$$|k_{g,\text{plane}}(\tfrac{\pi}{2})| = \frac{1}{|\cos\theta_0|}$$

- the geodesic curvature of α w.r.t the cylinder.

$\alpha'' = k_g\vec{n} + k_n\vec{N}$ where $\vec{N} = (\cos t, \sin t, 0)$

$$\therefore \quad k_n = \langle \alpha'', \vec{N} \rangle = -1$$

$$\therefore |k_g| = |\alpha'' - k_n\vec{N}| = |(0, 0, -\sin t \cdot \tan\theta_0)| = |\sin t \cdot \tan\theta_0|$$

$$\therefore |k_{g,\text{cylinder}}(0)| = 0$$

$$|k_{g,\text{cylinder}}(\tfrac{\pi}{2})| = |\tan\theta_0|.$$

2. "$\Rightarrow$" if after reparametrization, α is a geodesic then there is a strictly increasing function

$$\phi: R \to R$$

such that $\beta(s) = \alpha(\phi(s))$ is a geodesic.

$\therefore \ [\beta''(s)]^T = 0$

$\therefore \ [\alpha''(\tau)\cdot(\phi'(s))^2 + \alpha'(\tau)\cdot\phi''(s)]^T = 0$

$\therefore \ [\alpha''(\tau)]^T = -\dfrac{\phi''(s)}{[\phi'(s)]^2}\cdot\alpha'(\tau)$ ($\phi'(s) \neq 0$ because ϕ is strictly increasing)

$= \lambda(\tau)\cdot\alpha'(\tau)$

where $\lambda(\tau) = -\dfrac{\phi''(\phi^{-1}(\tau))}{[\phi'(\phi^{-1}(\tau))]^2}$ is a smooth function.

"$\Leftarrow$" if $[\alpha''(\tau)]^T = \lambda(\tau)\cdot\alpha'(\tau)$ for some smooth function $\lambda(\tau)$.

define $\psi(\tau) = \int e^{\int\lambda(\tau)}$

$\therefore$ $\psi(\tau)$ is a smooth strictly increasing function.

let $\phi = \psi^{-1}$, then let $s = \psi(\tau)$, $\tau = \psi^{-1}(s) = \phi(s)$

define $\beta(s) = \alpha(\phi(s))$

$\therefore$ By the previous computation,

$$\begin{aligned}
&[\beta''(s)]^T \\
&= [\alpha''(\tau)\cdot(\phi'(s))^2 + \alpha'(\tau)\cdot\phi''(s)]^T \\
&= [\alpha''(\tau)]^T\cdot\frac{1}{[\psi'(\tau)]^2} + \alpha'(\tau)\left[-\frac{\psi''(\tau)}{[\psi'(\tau)]^3}\right] \quad \text{since } \phi \text{ is the inverse of } \psi \\
&= [\alpha''(\tau)]^T\cdot\frac{1}{e^{2\int\lambda(\tau)}} + \alpha'(\tau)\cdot\left[-\frac{\lambda(\tau)\cdot e^{\int\lambda(\tau)}}{e^{3\int\lambda(\tau)}}\right] \\
&= [\alpha''(\tau)]^T\cdot\frac{1}{e^{2\int\lambda(\tau)}} - \lambda(\tau)\cdot\alpha'(\tau)\cdot\frac{1}{e^{2\int\lambda(\tau)}} \\
&= \frac{1}{e^{2\int\lambda(\tau)}}\left[[\alpha''(\tau)]^T - \lambda(\tau)\alpha'(\tau)\right] \\
&= 0
\end{aligned}$$

$\therefore$ $\beta(s) = \alpha(\phi(s))$ is a geodesic.

which means after reparametrization, α is a geodesic.

3、(a)

$$X_u = (-(a+r\cos v)\sin u,\ (a+r\cos v)\cos u,\ 0)$$
$$X_v = (-r\sin v\cdot\cos u,\ -r\sin v\cdot\sin u,\ r\cos v)$$

$$\therefore\ g_{ij} = \begin{bmatrix} (a+r\cos v)^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

$\therefore$ Use $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$, we have

$$\Gamma^k_{11} = \frac{1}{2}\begin{bmatrix} \frac{1}{(a+r\cos v)^2} & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}\left(2\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -2(a+r\cos v)r\sin v \end{bmatrix}\right)$$
$$= \begin{bmatrix} 0 \\ \frac{\sin v}{r}(a+r\cos v) \end{bmatrix}$$

$$\Gamma^k_{12} = \frac{1}{2}\begin{bmatrix} \frac{1}{(a+r\cos v)^2} & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2(a+r\cos v)r\sin v \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} -\frac{r\sin v}{a+r\cos v} \\ 0 \end{bmatrix}$$

$$\Gamma^k_{22} = \frac{1}{2}\begin{bmatrix} \frac{1}{(a+r\cos v)^2} & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}\left(2\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ the geodesic equation is

$$\begin{cases} u'' - \dfrac{2r\sin v}{a+r\cos v}\, u'v' = 0 \\[2mm] v'' + \dfrac{(a+r\cos v)\sin v}{r}\,(u')^2 = 0 \end{cases}$$

(b) Since $r(s)\cdot\sin\theta(s)$ is a constant for a geodesic on a revolution surface and α is tangent to $v=\frac{\pi}{2}$

$\therefore\ r(s)\cdot\sin\theta(s) = a\cdot 1 = a$

$\therefore\ r(s) \geq r(s)\cdot\sin\theta(s) = a$

$\therefore\ r\cos v + a \geq a$

$\cos v \geq 0$

$\therefore\ v \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

(c) let $\beta(t)=(a\cos t, a\sin t, r)$, $\beta'(t)=(-a\sin t, a\cos t, 0)$

$\therefore$ $\beta''(t)=(-a\cos t, -a\sin t, 0)$

$\because$ $\vec{N}(t)=(0,0,1)$

$\therefore$ $$k_g=\frac{\langle\beta'\times\beta'', \vec{N}\rangle}{|\beta'|^3}=\frac{a^2}{a^3}=\frac{1}{a}.$$

4. (a) $\because$ $g=\begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}$

$\therefore$ use $\Gamma_{ij}^k=\frac{1}{2}g^{kl}(\partial_i g_{jl}+\partial_j g_{il}-\partial_l g_{ij})$, we have

$$\Gamma_{11}^k=\frac{1}{2}\begin{bmatrix} \frac{1}{E} & 0 \\ 0 & \frac{1}{G} \end{bmatrix}\left(2\begin{bmatrix} E_u \\ 0 \end{bmatrix}-\begin{bmatrix} E_u \\ E_v \end{bmatrix}\right)$$

$$=\frac{1}{2}\begin{bmatrix} \frac{E_u}{E} \\ -\frac{E_v}{G} \end{bmatrix}$$

$$\Gamma_{12}^k=\frac{1}{2}\begin{bmatrix} \frac{1}{E} & 0 \\ 0 & \frac{1}{G} \end{bmatrix}\left(\begin{bmatrix} 0 \\ G_u \end{bmatrix}+\begin{bmatrix} E_v \\ 0 \end{bmatrix}-\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$=\frac{1}{2}\begin{bmatrix} \frac{E_v}{E} \\ \frac{G_u}{G} \end{bmatrix}$$

$$\Gamma_{22}^k=\frac{1}{2}\begin{bmatrix} \frac{1}{E} & 0 \\ 0 & \frac{1}{G} \end{bmatrix}\left(2\begin{bmatrix} 0 \\ G_v \end{bmatrix}-\begin{bmatrix} G_u \\ G_v \end{bmatrix}\right)$$

$$=\frac{1}{2}\begin{bmatrix} -\frac{G_u}{E} \\ \frac{G_v}{G} \end{bmatrix}$$

the geodesic equation is

$$\begin{cases} u''+\frac{E_u}{2E}(u')^2+\frac{E_v}{E}u'v'-\frac{G_u}{2E}(v')^2=0 \\ v''-\frac{E_v}{2G}(u')^2+\frac{G_u}{G}u'v'+\frac{G_v}{2G}(v')^2=0 \end{cases}$$

(b) when $E = G = e^{2f}$

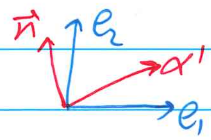
$$E_u = e^{2f}\cdot 2f_u \,, \qquad E_v = e^{2f}\cdot 2f_v$$

$$\therefore\ \Gamma_{11}^k = \frac{1}{2}\begin{bmatrix} \frac{E_u}{E} \\ -\frac{E_v}{G} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2f_u \\ -2f_v \end{bmatrix} = \begin{bmatrix} f_u \\ -f_v \end{bmatrix}$$

$$\Gamma_{12}^k = \frac{1}{2}\begin{bmatrix} \frac{E_v}{E} \\ \frac{G_u}{G} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2f_v \\ 2f_u \end{bmatrix} = \begin{bmatrix} f_v \\ f_u \end{bmatrix}$$

$$\Gamma_{22}^k = \frac{1}{2}\begin{bmatrix} -\frac{G_u}{E} \\ \frac{G_v}{G} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2f_u \\ 2f_v \end{bmatrix} = \begin{bmatrix} -f_u \\ f_v \end{bmatrix}$$

5. (a) $\vec{a} = \vec{N} \times \alpha'$



$$= (e_1 \times e_2) \times (\cos\theta \cdot e_1 + \sin\theta \cdot e_2)$$
$$= \cos\theta \cdot (e_1 \times e_2) \times e_1 + \sin\theta\, (e_1 \times e_2) \times e_2$$
$$= \cos\theta \cdot e_2 - \sin\theta \cdot e_1$$

(b) $\alpha'' = k_g \vec{n} + k_n \cdot \vec{N}$ where $\vec{n} = \vec{N} \times \frac{\alpha'}{|\alpha'|} = -\sin\theta \cdot e_1 + \cos\theta \cdot e_2$

$$\therefore\ k_g = \langle \alpha'', \vec{n} \rangle$$
$$= \langle e_1' \cos\theta - e_1 \sin\theta \cdot \theta' + e_2' \sin\theta + e_2 \cos\theta \cdot \theta',\ -\sin\theta\, e_1 + \cos\theta\, e_2 \rangle$$
$$= \sin^2\theta \cdot \theta' - \sin^2\theta \langle e_2', e_1 \rangle + \cos^2\theta \langle e_1', e_2 \rangle + \cos^2\theta \cdot \theta'$$

$(\langle e_1', e_1 \rangle = \langle e_2', e_2 \rangle = \langle e_1, e_2 \rangle = 0)$

$$= \sin^2\theta \langle e_2, e_1' \rangle + \cos^2\theta \langle e_1', e_2 \rangle + \theta'$$
$$= \langle e_1', e_2 \rangle + \theta'$$
$$= \langle (e^{-f} X_1)', e^{-f} \cdot X_2 \rangle + \theta'$$
$$= \langle e^{-f} \cdot (X_1)', e^{-f} X_2 \rangle + \theta' \qquad (\langle X_1, X_2 \rangle = 0)$$
$$= e^{-2f} \langle (X_1)', X_2 \rangle + \theta'$$
$$= e^{-2f} \langle u' X_{11} + v' X_{12}, X_2 \rangle + \theta'$$
$$= e^{-2f} \left(u' \langle X_{11}, X_2 \rangle + v' \langle X_{12}, X_2 \rangle \right) + \theta'$$
$$= e^{-2f} \left(u' \langle h_{11} \vec{N} + \Gamma_{11}^k X_k, X_2 \rangle + v' \langle h_{12} \vec{N} + \Gamma_{12}^k X_k, X_2 \rangle \right) + \theta'$$
$$= e^{-2f} \left(u' e^{2f} \cdot \Gamma_{11}^2 + v' e^{2f} \Gamma_{12}^2 \right) + \theta'$$
$$= u' \Gamma_{11}^2 + v' \Gamma_{12}^2 + \theta'$$
$$= (-u' f_v + v' f_u) + \theta' \qquad (\text{by Q4, (b)})$$